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On the finiteness of nonderivative nonpolynomial Lagrangians

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Abstract. It is shown how to achieve a finite (and unitary and causal) S matrix for a large class of nonderivative nonpolynomial Lagrangians.

1. Introduction

It has been suggested that nonpolynomially interacting quantum fields give rise to S matrix elements which do not contain the standard high energy divergences of polynomially interacting fields. The particularly simple exponential interaction $G\,\mathrm{e}^{\lambda\phi}$ of a scalar field ϕ and related interactions have been much discussed recently, particularly in the second order in the major coupling constant G. From this it has been conjectured (Salam 1971) that the exponential interaction will always damp out the high energy divergences of any polynomial multiplying it in the interaction. We wish to discuss this conjecture here and more generally to determine as large a class as possible of nonpolynomial interactions which give finite, unitary and analytic S matrix elements to all orders in the interaction.

In a previous paper (Taylor 1971a, to be referred to as I) one of us showed how to construct an S matrix finite, unitary and causal to all orders in the major coupling constant for the exponential interaction with zero mass. Here we extend this result to a large class of nonderivative massless nonpolynomial Lagrangians. Our class includes both localizable interactions, like the exponential, for which the results can be said to be causal, and nonlocalizable, for which the definition of causality is more difficult (Taylor 1971b).

In § 2 we summarize the work on the exponential interaction published before, then the results of I.

In § 3 we summarize the work on other interactions, in particular discussing the condition of localizability in lowest order.

In § 4 we give the S matrix for a general Lagrangian as a superposition of S matrices for exponential interactions. This allows us to derive conditions on the Lagrangian for finiteness, unitarity and causality of the S matrix.

In § 5 we consider the special cases $\phi^n\,\mathrm{e}^{\lambda\phi}$, ϕ^n and various interactions with several fields. We find that the presence of the exponential is crucial for our results, so verifying Salam's conjecture.

In § 6 we discuss the problem of the selfmass, which is particularly difficult in the massless case owing to the infrared divergence. We describe a solution.

2. Previous discussions

The two point function has been given by many people (Okubo 1954, Volkov 1968, Delbourgo *et al* 1969, Lehmann and Pohlmeyer 1971, Mitter 1970). The methods vary: the 'superpropagator' $e^{\lambda^2\Delta}$, $\Delta = \{i(\Box+m^2)\}^{-1}$, may undergo several operations—an ϵ inserted, Fourier transform taken, λ moved to the imaginary axis, and for momenta moved to the euclidean region then have appropriate limits taken and continuations made.

All effectively decide the values of the otherwise arbitrary constants one obtains by leaving out the divergent parts of Δ^n in

$$\sum_n \frac{\lambda^{2n}}{n!}\Delta^n$$

and so give unitarity and analyticity provided the part thrown away is imaginary—it may be necessary to arrange this for example by taking a certain weighted average of different continuations back to real λ. Lehmann and Pohlmeyer (1971) have shown that among the sets of values given by the various methods there is one such that the rate of growth of the amplitude for space-like momenta is minimal with respect to all possible sets of values.

The three point function has received some special attention from Daniell and Mitter (1971), and Pohlmeyer (1972). They show the existence of a minimal singularity prescription compatible with unitarity and causality.

For higher orders definitions of S matrix elements have been given by Efimov (1965), Salam and Strathdee (1970) and Fukuda (1970). Efimov's results are cut-off dependent and nonlocal, and the cut-off is arbitrary. Due to these defects we will not consider his method further here. Salam and Fukuda both give formal discussions with no attempt to prove finiteness for physical values of the momenta.

In I, the S operator is given by

$$S = \sum_{N\geqslant 0} \frac{(iG)^N}{N!} : \prod_{i=1}^{N} \int dx_i \, e^{\lambda\phi(x_i)} : S(x_1 \dots x_N)$$

where

$$S(x_1 \dots x_N) = \lim_{\epsilon\downarrow 0} S^\epsilon(x_1 \dots x_N).$$

$S^\epsilon(x_1 \dots x_N)$ is the value of $S^\epsilon_\gamma(x_1 \dots x_N)$ at $\gamma = 1$, continued from $\gamma > \frac{5}{2}$ where

$$\tilde{S}^\epsilon_\gamma(p_1 \dots p_N) = \prod_{i<j}^{N} \frac{-1}{32\pi} \int \frac{dp_{ij}}{(2\pi)^4} \int \frac{dz_{ij}\lambda^{2z_{ij}}\{-(p_{ij}^2+i\epsilon)/16\pi^2\}^{z_{ij}-2}}{\tan \pi z_{ij}\Gamma(z_{ij}-1)\Gamma(z_{ij})\Gamma(z_{ij}+1)\sin \pi\gamma z_{ij}}$$
$$\times \prod_{i=1}^{N} (2z)^4 \delta\left(p_i - \sum_j p_{ij}\right) \tag{1}$$

where $p_{ij} = -p_{ji}$, the z_{ij} contour is parallel to the imaginary axis between 0 and -1.

Continued forms of unitary and causality were derived for $\tilde{S}^\epsilon_\gamma(p_1 \dots p_N)$ and it was shown in I that these could then be continued back to give unitarity and causality for the physical S matrix elements. The conjecture of Lehmann (1971) concerning the existence of a minimal singularity prescription for all orders has not yet been proved.

Let us now consider the general nonpolynomial interaction

$$L_{\mathrm{int}}(\phi) = \sum_{n \geqslant 0} \frac{a_n \phi^n}{n!}. \tag{2}$$

The 2-point function $G(x_1 x_2)$ in second order in the interaction is

$$G(x_1 x_2) = \sum_{n \geqslant 0} \frac{a_{n+1}^2}{n!}(n+1)\Delta^{n+1}(x_1 - x_2).$$

In momentum space, for large P^2, the regularized version of G obtained for example by using Mellin integral techniques, behaves as

$$\tilde{G}(P) \sim \sum_{n \geqslant 0} \frac{a_{n+1}^2}{n!}(n+1)\frac{P^{2(n-1)}}{n!n+1!}.$$

In order that G is a generalized function that is localizable it is necessary that $\tilde{G}(P)$ increase for large P^2 slower than $\exp(\sqrt{|P^2|})$. This requires that a_n has exponential growth of order $\frac{1}{2}\alpha$ at infinity with $\alpha < 1$. If α is larger than 1 the theory is nonlocalizable.

The discussion in I proved finiteness unitarity and causality of all the higher order interactions, under the same restrictions on the order of growth of the coefficients a_n.

3. Conditions for finiteness

It appears most suitable to consider the interaction Lagrangian in the form used in I, that is, as a superposition of exponentials. We take instead of (2) the expression

$$L_{\mathrm{int}}(\phi) = :\int \mathrm{d}\lambda \rho(\lambda)\, \mathrm{e}^{\lambda\phi}:$$

where

$$a_n = \int \mathrm{d}\lambda \rho(\lambda)\lambda^n. \tag{3}$$

The discussion in I showed that unitarity and causality will be automatically satisfied by a large class of weight functions ρ. Let us now discuss in detail the conditions on ρ for finiteness. The expression (1) has to be modified in this case by replacing the factor $\lambda^{2z_{ij}}$ by $(\lambda_i\lambda_j)^{z_{ij}}$ and then multiplying by $\Pi_{i=1}^{N}\int \rho(\lambda_i)\,\mathrm{d}\lambda_i$. The integrand in the resulting expression, regarded as a function of the variables $\lambda_1 \ldots \lambda_N$ is not analytic. It is possible to evaluate in detail the singularity structure. This arises from the residues of the modified integrand in (1) on reforming the z_{ij} contours to encircling the non-negative integers. These residues, as can be seen by inspection of (1), will contain non-negative integral powers of the λ_i, multiplied by powers of $\ln \lambda_i$. The power of each λ_i that arises in the modified version of (1), is just

$$w_i = \sum_j z_{ij}.$$

There are at least two lines that enter i. For each of these, only residues for $z_{ij} \geqslant 1$ can arise, thus the power of λ_i that arises in a particular residue is always at least two; there may also be a power of $\ln \lambda_i$ present. This is clearly seen in the two point function in lowest order, which has a $\lambda^2 \ln \lambda$ behaviour. In order that the integral over the variables λ_i be well defined it is necessary that $\rho(\lambda)$ be not too singular at $\lambda = 0$. The essential property that must be satisfied is that for $r = 0, 1, 2$ and any s

$$\left| \int_0 \mathrm{d}\lambda \rho(\lambda) \lambda^r (\ln \lambda)^s \right| < \infty. \tag{4}$$

This condition is thus the one we impose on weight functions.

It is evident that the only other point at which integration over λ can produce difficulties is $\lambda = \infty$. We know that the pure exponential interaction has bounds

$$|\tilde{S}(p_1 \ldots p_N)| \lesssim c \exp(\max_{ij} |\lambda^2 p_i p_j|^{1/3}) \tag{5}$$

where c is some constant, after exclusion of intermediate single particle poles.

When we use the interaction (3), we have to evaluate

$$\left| \prod_i \int \mathrm{d}\lambda_i \rho(\lambda_i) \tilde{S}(\lambda_1 \ldots \lambda_N, p_1 \ldots p_N) \right|$$

and the bound (5) becomes

$$c \left| \prod_i \int \mathrm{d}\lambda_i \rho(\lambda_i) \exp(\max_{ij} |\lambda_i \lambda_j p_i p_j|^{1/3}) \right|. \tag{6}$$

In order for this to be finite, we impose the condition that $\rho(\lambda)$ decreases for large λ faster than $\exp(-\lambda^{2/3})$, in other words is an element of the function space S_α, for any $\alpha < \frac{3}{2}$, for large λ. Thus we require

$$\rho(\lambda) \in G'_\alpha$$

where

$$G_\alpha = \{\phi(\lambda) : |\phi(\lambda)| < \exp(\lambda^{1/\alpha}) \text{ as } \lambda \to \infty\}. \tag{7}$$

The conditions, then, on ρ that are finally arrived at, are (4) and (7).

Let us consider briefly the class of ρ that give localizability. For ρ satisfying (7), we find using (3)

$$\begin{aligned} |a_n| &\leqslant \int_0^\infty \mathrm{d}\lambda \lambda^n \exp(-\lambda^{1/\alpha}) \\ &\propto \Gamma(n\alpha + \alpha) \end{aligned}$$

so that we certainly have localizability if $\alpha < \frac{1}{2}$. It is to be expected that localizability fails if $\alpha > \frac{1}{2}$; the case $\rho(\lambda) = \mathrm{e}^{-\sqrt{\lambda}}$ gives $a_n = \Gamma(\frac{1}{2}n + \frac{1}{2})$, which is on the verge. Thus we see that the finiteness condition still allows a large case of nonlocalizable interactions to be discussed including the simple case of rational functions of ϕ. We see this by a partial fraction expression with $\rho(\lambda) = \mathrm{e}^{-\lambda}$ for the interaction $1/(1-\lambda\phi)$.

4. Special cases

Let us turn first to the interactions conjectured to be finite to all orders by Salam. These are $\phi^n \mathrm{e}^{\lambda_0 \phi}$, for $n = 0, 1, 2, \ldots$. The corresponding weight functions ρ are $\delta^{(n)}(\lambda - \lambda_0)$ which as we see satisfy both conditions (4) and (7) of the previous section.

If we wish to return to the polynomial domain by letting λ_0 go to zero we meet a violation of condition (4) as soon as $n \geqslant 2$, due to the fact that

$$\left| \int_0 \mathrm{d}\lambda \delta^{(n)}(\lambda - \lambda_0)\lambda^2 \ln \lambda \right| \to \infty$$

as $\lambda_0 \to 0$ if $n \geqslant 2$. In this way we meet again the standard ultraviolet divergences of polynomial quantum field theories. We note in passing that $\mathrm{e}^{\lambda\phi} - 1 - \lambda\phi$ is thus finite to all orders, but $\mathrm{e}^{\lambda\phi} - 1 - \lambda\phi - \frac{1}{2}\lambda^2\phi^2$ is not, though this latter is in lowest order.

When we turn to consider several fields, we see that the exponential of one field can be used to damp out ultraviolet divergences arising from a polynomial interaction in another. Thus, let us consider the interaction Lagrangian $\chi^n \mathrm{e}^{\kappa\phi}$ where χ and ϕ are both massless. This has the exponential representation

$$\int_0^\infty \mathrm{d}\lambda \rho(\lambda) : \mathrm{e}^{\lambda\chi + \kappa\phi} : \tag{8}$$

where $\rho(\lambda) = \delta^{(n)}(\lambda)$. The superpropagator in this case is

$$\langle T\{\mathrm{e}^{\lambda\chi_1 + \kappa\phi_1}, \mathrm{e}^{\lambda\chi_2 + \kappa\phi_2}\}\rangle_0 = \mathrm{e}^{(\lambda^2 + \kappa^2)\Delta_{12}}. \tag{9}$$

The higher order terms arising from (8) involve replacing λ^2 in (9) by $\lambda_i\lambda_j$ so that we have, on taking residues in the z_{ij} variables, expressions of the form

$$\int\int (\kappa^2 + \lambda_i\lambda_j)^{m_{ij}} |\ln(\kappa^2 + \lambda_i\lambda_j)|^{n_{ij}} \rho(\lambda_i)\rho(\lambda_j)\, \mathrm{d}\lambda_i\, \mathrm{d}\lambda_j.$$

In spite of the delta function behaviour of ρ, all these expressions are finite, due to the presence of κ^2. This interaction of the form (8) gives finite results to all orders provided $\rho(\lambda)$ satisfies condition (7) for large λ and is a distribution in λ inside a bounded region. This evidently includes the interactions $\chi^n \mathrm{e}^{\kappa\phi}$ we started with.

5. Selfmass

We have taken a general interaction and assumed that the physical mass is zero. This of course is only so under certain conditions on the Lagrangian, those such as make

$$S|p\rangle = |p\rangle$$

for all single particle states $|p\rangle$.

In first order in the major coupling constant this requires that the interaction contain no mass term. An example of such an interaction belonging to the class given by (7) is

$$G\, \mathrm{e}^{\lambda\phi} - \tfrac{1}{2}G\lambda\phi\, \mathrm{e}^{\lambda\phi}.$$

Any interaction satisfying (7) can be given zero selfmass in first order by a similar subtraction.

In higher orders we have an additional problem, that of infrared divergence. This difficulty is basically that arising from the term proportional to $\ln(-P^2)$ in the single particle Green function, this being infinite at $P^2 = 0$. This is not present in the massive case, when the term becomes $\ln(4m^2 - P^2)$, being finite at $P^2 = m^2$. This term corresponds to the logarithmically divergent terms in the nonregularized perturbation expression, that is, the expansion in powers of both major and minor coupling constants.

There can be many higher order terms that contribute. These are absent if there are no ϕ^2 and ϕ^3 terms in the interaction Lagrangian. To see this, take any diagram with N vertices, L internal lines, and two external lines. The degree of divergence of such a graph will be

$$d = 4(L-N+1)-2L = 2L-4N+4.$$

We assume that at each vertex there are at least four lines, so that $4N \leqslant 2L+2$. Thus $d \geqslant 2$. This will correspond to the absence of the term $\ln(-\lambda^2 P^2)$ in the regularized version, but of course allowing terms $(\lambda^2 P^2)^r \{\ln(-\lambda^2 P^2)\}^s$ for $r \geqslant 2$. These terms are zero at $P^2 = 0$, as required for stability of single particle states.

We can remove the ϕ^3 term by a subtraction similar to that which removed the ϕ^2 term, and in various ways. An example of a suitably subtracted interaction derived from the exponential is

$$e^{\lambda\phi}\{1-\tfrac{1}{2}\lambda\phi+\tfrac{1}{12}(\lambda\phi)^3\}.$$

A simpler one is

$$\lambda\phi^n \, e^{\lambda\phi}$$

for any $n \geqslant 4$. The particular Lagrangian chosen will depend on further physical conditions.

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